An Introduction to the Noncommutative Sphere and some Extensions

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Abstract

This is a copy of the talk given at the conference "Methods in Field Theory" at Stará Lesná, The Slovak Republic. Sepemeber 22-26, 1997. An introduction to the noncommutative sphere and a summary of the results of articles q-alg/9703038 [1], and q-alg/9708003 [2] is given. This includes results about the the algebra of scalar, spinor and vector fields on the noncommutative sphere. Possible extensions of these results including a "Wick rotation" to the one and two sheeted hyperboloid are also examined.

Contents

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1 Introduction

Noncommutative geometry is, at present, similar to string theory during the 80's. If you ask five noncommutative geometers what they mean by the subject you will get at least half a dozen different answers. Here, at least for me, is a working definition:

> Almost all the tools[1] used to study ordinary differential geometry can be expressed in terms of the functions from a base manifold to $\mathbb{R}$ or $\mathbb{C}$. The algebra of these functions using pointwise multiplication is commutative. The object of Noncommutative Geometry is to replace this algebra with a noncommutative algebra, and then to find and interpret analogues of these tools.

The rest of noncommutative geometry is, in my opinion, a wish list. These include:

(1) One would like a one parameter algebra $\mathcal{A}(\hbar)$, such that $\mathcal{A}(\hbar = 0) = C(\mathcal{M})$, the commutative algebra of functions on the manifold with pointwise multiplication, and such that the commutator is of order $\hbar$. I.e. $[f, g] = fg - gf = O(\hbar)$.

(2) If $\mathcal{M}$ has a Poisson structure then one would like the antisymmetric part of the first order correction to agree with the Poisson bracket. $\{f, g\} = \lim_{\hbar\to 0}(\frac{1}{\hbar}[f, g])$. In this sense it is similar to deformation quantisation.

(3) A definition of vector fields, and covector fields over $\mathcal{M}$.

(4) A definition of exterior derivative and corresponding exterior calculus.

(5) A definition of intergration over the manifold, and thus a Hilbert space structure for $C(\mathcal{M})$

(6) Spinor fields if $\mathcal{M}$ admits a spinor structure.

(7) If $\mathcal{M}$ is to be thought of as spacetime we need to define objects such as metrics, connections, gauge transformations, curvature, gravity, etc.

(8) If considering manifolds in general, one would like to extend the theory of algebraic topology. Noncommutative geometry may provide a mechanism for topology change.

In this talk I hope to give an my personal interpretation of the noncommutative or "fuzzy" sphere. The idea is to give the reader some motivation for looking at my work. I shall only state the theorems and the reader is invited to look at my papers for the details. There are many reasons for studying the sphere.

(1) It is simple enough that most of the results can be explicitly worked out, and being two dimensional and compact, it can also be pictured.

(2) The space of functions on the sphere is a representation of $su(2)$. As the simplest of classical groups it has been studied in depth and there are many results which can be used.

(3) The sphere has curvature.

[1] One can generate the original manifold from the algebra of functions on that manifold only if the manifold is algebraic or holomorphic. However the detailed differential structure of a manifold is seldom of interest to physicists who usually assume at least a C^∞ structure.

(4) The results and insight gained from the study of the sphere can be extended to certain symmetric homogeneous spaces (co-adjoint orbits) of higher dimensions, and to non compact spaces.

(5) It, together with the torus, can be used to study the vector space, connections and other tools used in differential geometry and general relativity.

(6) It, and the torus are the only 2-dimensional manifolds for which noncommutative analogues are "easy".

2 The Noncommutative Sphere

In paper [1] details are given about the noncommutative analogue of the space of functions on the sphere. This satisfies (1) and (2) of the wish list.

2.1 The commutative sphere: spherical harmonics

As mentioned in the introduction, we must first consider the algebra, $C(S^2)$, of complex valued functions from the sphere. Ignoring problems to do with linear analysis[2], we can write the functions on the sphere as sums of *Spherical Harmonics* Y_n^m, where $n = 0, 1, 2, \dots$ and $m = -n, -n+1, \dots, n$. In spherical coordinates (θ, ϕ), where $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$ we can write the spherical harmonics in terms of Legendre polynomials:

$$Y_n^m(\theta,\phi) = \left(\frac{(n+m)!(2n+1)}{(n-m)!}\right)^{1/2} e^{-im\phi} P_n^{-m}(\cos\theta)$$

Alternatively one defines them in terms of a subset of the rotation matrices.

$$Y_n^{-m}(\beta,\alpha) = \left(\frac{2n+1}{4\pi}\right)^{1/2} D^n_{m0}(\alpha,\beta,\gamma) \tag{2.1}$$

where (α, β, γ) are the Euler angles. (The Spherical Harmonic is, of course, independent of the value of γ.)

For our purposes, there is an alternative definition of the spherical harmonics given in terms of cartesian coordinates on $\mathbb{R}^3$. For a sphere of radius R we define

$$f_{\text{polar}}(\theta,\phi) = f_{\text{cart}}(x,y,z) \qquad \text{where} \qquad x = R\cos\phi\,\sin\theta\,,\ y = R\sin\phi\,\sin\theta\,,\ z = R\cos\theta$$

It is clear that for any $f_{\text{polar}}(\theta,\phi)$ there will be many functions $f_{\text{cart}}(x,y,z)$. For example the functions

$$f_{\text{cart}}(x,y,z) = x^2+y^2+z^2 \qquad \text{and} \qquad f_{\text{cart}}(x,y,z) = R^2$$

represent the same function on S^2. To choose a unique way of writing the functions on the sphere we require the polynomial to be *symmetric* and *formally tracefree*. That is there is a unique way of writing any[3] function $f : S^2 \mapsto \mathbb{C}$ as

$$f(x_1,x_2,x_3) = f_0 + \sum_{a=1}^{3} f_a x_a + \sum_{a,b=1}^{3} f_{ab} x_a x_b + \dots + \sum_{a_1,\dots a_n=0}^{3} f_{a_1 a_2 \cdots a_n} x_{a_1} x_{a_2} \cdots x_{a_n} + \cdots \tag{2.2}$$

where

[2] We avoid the problems of linear analysis by considering only the algebra of finite sums of spherical harmonics.

[3] Here any function means any function which is a nicely convergent sum of spherical harmonics

(1) $f_{a_1 a_2 \cdots a_n}$ is totally symmetric in any two indices. and

(2) $\sum_{b=1}^{3} f_{b b a_3 \cdots a_n} = 0$

The subspace $\mathcal{P}^n$ of polynomials of order n

$$\mathcal{P}^n = \mathrm{span}\{f_{a_1 a_2 \cdots a_n} x_{a_1} x_{a_2} \cdots x_{a_n}\}$$

which satisfy the conditions above, has dimension $2n+1$. This space is spanned by the spherical harmonics

$$Y_n^{-n},\ Y_n^{-n+1}, \ldots, Y_n^0, \ldots, Y_n^{n-1},\ Y_n^n,$$

The low order spherical harmonics are given in both polar coordinates and cartesian coordinates in Table 1. Here $J_+ = x + iy$ and $J_- = x - iy$.

2.2 The commutative sphere: the inner product

There is a natural inner product on $C(S^2)$ given by

$$\langle f, g\rangle_{S^2} = \frac{1}{4\pi R^2}\int_{S^2} \overline{f} g \sin\theta d\phi d\theta$$

which satisfies

$$\frac{1}{4\pi R^2}\int Y_n^m(\theta,\phi)\sin\theta d\phi d\theta = \begin{cases} 1 & m = 0 \text{ and } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

therefore we can define the inner product as simply

$$\langle f, g\rangle_{S^2} = \pi_0(\overline{f} g)$$

where $\pi_0(f)$ means take the Y_0^0 coefficient of the f, or if written as (2.2), take the f_0 component.

2.3 The noncommutative algebra $\mathcal{P}(\varepsilon, R)$

We construct a two parameter algebra $\mathcal{P}(\varepsilon, R)$ which may be thought of as the noncommutative analogue of $C(S^2)$ as follows: The universal enveloping algebra of $su(2)$ is given by

$$\mathcal{U}(\varepsilon) = \{\text{Free noncommuting algebra of polynomials in } x, y, z\ \}\Big/ \sim$$

where x, y, z obey the commutation relations for $su(2)$

$$[x, y] \sim i\varepsilon z,\ \ [y, z] \sim i\varepsilon x,\ \ [z, x] \sim i\varepsilon y \tag{2.3}$$

The Casimir is of course given by $x^2 + y^2 + z^2$. This belongs to the center of $\mathcal{U}(\varepsilon)$. We may therefore quotient $\mathcal{U}(\varepsilon)$

$$\mathcal{P}(\varepsilon, R) = \mathcal{U}(\varepsilon)\Big/ J(R)$$

where J(R) is the two sided ideal generated by

$$x^2 + y^2 + z^2 \sim R^2 \tag{2.4}$$

This gives an infinite dimensional two parameter algebra $\mathcal{P}(\varepsilon, R)$ where in the most general case, $\varepsilon, R \in \mathbb{C}$. For different values of ε and R we obtain:

Harmonic	Polar coordinates (for $\varepsilon = 0$)	Formally traceless symmetric polynomial ($\forall\varepsilon$)	Hahn Polynomial ($\forall\varepsilon$)
P_0^0	1	1	1
P_1^1	$Re^{-i\phi}\sin\theta$	J_+	J_+
P_1^0	$-\sqrt{2}R\cos\theta$	$-\sqrt{2}z$	$-\sqrt{2}z$
P_1^{-1}	$-Re^{i\phi}\sin\theta$	$-J_-$	$-J_-$
P_2^2	$R^2e^{-2i\phi}\sin^2\theta$	J_+^2	J_+^2
P_2^1	$-2R^2e^{-i\phi}\cos\theta\sin\theta$	$-2(J_+z + zJ_+)$	$-J_+(2z+\varepsilon)$
P_2^0	$R^2\sqrt{2/3}(3\cos^2\theta - 1)$	$\sqrt{2/3}(2z^2 - J_+J_- - J_-J_+)$	$\sqrt{2/3}(3z^2 - R^2)$
P_2^{-1}	$2R^2e^{i\phi}\cos\theta\sin\theta$	$2(J_-z + zJ_-)$	$J_-(2z-\varepsilon)$
P_2^{-2}	$R^2e^{2i\phi}\sin^2\theta$	J_-^2	J_-^2
P_3^3	$R^3e^{-3i\phi}\sin^3\theta$	J_+^3	J_+^3
P_3^2	$-R^3e^{-2i\phi}\sqrt{6}\cos\theta\sin^2\theta$	$-\sqrt{6}(J_+^2z + J_+zJ_+ + zJ_+^2)$	$-\sqrt{6}J_+^2(z+\varepsilon)$
P_3^1	$R^3e^{-i\phi}\sqrt{3/5}\sin\theta(5\cos^2\theta - 1)$	$-\sqrt{3/5}(J_+^2J_- + J_+J_-J_+ + J_-J_+^2)$ $+4\sqrt{3/5}(J_+z^2 + zJ_+z + z^2J_+)$	$\sqrt{3/5}J_+(5z^2 + 5\varepsilon z + 2\varepsilon^2 - R^2)$
P_3^0	$R^3(2/\sqrt{5})(3\cos\theta - 5\cos^3\theta)$	$(2/\sqrt{5})(z^3 - J_+J_-z - J_+zJ_- - J_-J_+z$ $-J_-zJ_+ - zJ_+J_- - zJ_-J_+)$	$(-2/\sqrt{5})z(5z^2 - 3R^2 + \varepsilon^2)$
P_3^{-1}	$-R^3e^{i\phi}\sqrt{3/5}\sin\theta(5\cos^2\theta - 1)$	$\sqrt{3/5}(J_-^2J_+ + J_+J_+J_- + J_+J_-^2)$ $-4\sqrt{3/5}(J_-z^2 + zJ_-z + z^2J_-)$	$-\sqrt{3/5}J_-(5z^2 - 5\varepsilon z + 2\varepsilon^2 - R^2)$
P_3^{-2}	$-R^3e^{2i\phi}\sqrt{6}\cos\theta\sin^2\theta$	$-\sqrt{6}(J_-^2z + J_-zJ_- + zJ_-^2)$	$-\sqrt{6}J_-^2(z-\varepsilon)$
P_3^{-3}	$e^{3i\phi}R^3\sin^3\theta$	$-J_-^3$	J_-^3

Table 1: Low order Spherical harmonics written in both polar and traceless cartesian form (the normalisation may not be conventional)

(1) The commutative algebra of finite sums of harmonics on the sphere (when $\varepsilon = 0$ and $R \in \mathbb{R}$, $R > 0$). In this case R is radius of the sphere.

(2) The finite matrix representation of $su(2)$. When $\varepsilon^2(N^2-1) = 4R^2$ and $N \in \mathbb{Z}$, $N \geq 1$, then $M_N(\mathbb{C})$ forms a quotient algebra, and R^2 is the Casimir operator.

(3) A noncommutative algebra of polynomials which is an infinite dimensional reducible but non decomposable representation of $su(2)$, for all values of $\varepsilon \neq 0$.

As mentioned, for the case $\varepsilon = 0$, the noncommutative algebra $\mathcal{P}(\varepsilon = 0, R)$ becomes the commutative algebra $C(S^2)$. The first correction term in the product of two functions is given by the Poisson bracket on S^2, defined with respect to its natural measure[4].

$$\begin{aligned}
&\{\bullet, \bullet\} : C_{00}(S^2) \times C_{00}(S^2) \mapsto C_{00}(S^2) \\
&\{f, g\} = \lim_{\varepsilon \to 0} \left(\frac{1}{i\varepsilon} [f, g] \right) \\
&\{f, g\} = \frac{1}{R \sin\theta} \left(\frac{\partial f}{\partial \phi} \frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \phi} \right)
\end{aligned}$$

It is still unknown (at least to me) the exact nature of the higher order correction terms, and whether or not they form a $\star$-product in the sense of Flato et al. I am looking into this problem.

We can still represent the functions using (2.2). Thus we still have a sesquilinear form on $\mathcal{P}(\varepsilon, R)$ given by

$$\langle f, g \rangle = \pi_0(f^\dagger g)$$

where $\pi_0(f)$ means take the f_0 component of f, and $f^\dagger$ is the hermitian conjugate of f defined by:

$$\dagger : \mathcal{P} \mapsto \mathcal{P},\ (ab)^\dagger = b^\dagger a^\dagger,\ x^\dagger = x,\ y^\dagger = y,\ z^\dagger = z,\ \lambda^\dagger = \overline{\lambda} \qquad \text{for } \lambda \in \mathbb{C}$$

However this sesquilinear form is no longer positive definite since $\pi_0(f, f)$ may be positive, negative or zero. (See (2.8) below.)

2.4 P_n^m an orthogonal basis for $\mathcal{P}$

There is a basis[5] of $\mathcal{P}(\varepsilon, R)$ given by

$$\{P_n^m \mid n, m \in \mathbb{Z}, n \geq 0, |m| \leq n\} \tag{2.5}$$

where (for $\varepsilon \neq 0$)

$$P_n^m = \varepsilon^{m-n} \left(\frac{(n+m)!}{(2n)!\,(n-m)!} \right)^{1/2} \left(\mathrm{ad}_{J_-}\right)^{n-m} \left(J_+{}^n\right) \tag{2.6}$$

where $J_\pm = x \pm iy$ and $\mathrm{ad}_{J_-} f = [J_-, f]$. This basis is orthogonal with respect to the sesquilinear form

$$\langle P_{n_1}^{m_1}, P_{n_2}^{m_2} \rangle = \delta_{m_1 m_2} \delta_{n_1 n_2} \|P_{n_1}^{m_1}\|^2 \tag{2.7}$$

[4]It is necessary to map $C(S^2) \mapsto \mathcal{P}(R, \varepsilon)$ in order to take the commutator. This map is well defined.

[5]In [1] P_n^m are normalised by using a constant α_n. Also we use κ in place of ε.

where the norm of P_n^m is independent of m and is given by

$$\|P_n^m\|^2 = \frac{(n!)^2}{(2n+1)!}\prod_{r=1}^{n}(4R^2 + \varepsilon^2(1-r^2)) \tag{2.8}$$

We can see that $\|P_n^m\|^2$ may be positive, negative, or zero depending on the values od R, ε and n.

Each P_n^m can be written as a homogeneous formally tracefree symmetric polynomial in (x, y, z) of order n. As such they are independent of the value of ε and R. Therefore they are exactly the same as the spherical harmonics when written in this form, given in table 1.

Each P_n^m is an eigenvector of the operators ad_z and $\Delta = \mathrm{ad}_z^2 + \mathrm{ad}_{J_+}\mathrm{ad}_{J_-} + \mathrm{ad}_{J_-}\mathrm{ad}_{J_+}$.

$$\mathrm{ad}_z P_n^m = \varepsilon m P_n^m \tag{2.9}$$

$$\Delta P_n^m = \varepsilon^2 n(n+1) P_n^m \tag{2.10}$$

The ladder operators $\mathrm{ad}_{J_+}, \mathrm{ad}_{J_-}$ increase and decrease m so that $\mathcal{P}^n$ can be viewed as a $2n+1$ dimensional adjoint representation of $su(2)$.

$$\mathrm{ad}_{J_+} P_n^m = \varepsilon (n-m)^{1/2}(n+m+1)^{1/2} P_n^{m+1} \tag{2.11}$$

$$\mathrm{ad}_{J_-} P_n^m = \varepsilon (n+m)^{1/2}(n-m+1)^{1/2} P_n^{m-1} \tag{2.12}$$

The effect of taking the hermitian conjugate is given by

$$(P_n^m)^\dagger = (-1)^m P_n^{-m} \tag{2.13}$$

The formula for the combination of 2 basis elements may be obtained by substituting $r_1 = r_2 = r = 0$ and $\varepsilon^2(k - \frac{1}{2})^2 = R^2 + \frac{1}{4}$ into (4.2) below.

2.5 Alternative ways of writing $\mathcal{P}$ and P_n^m

As well as writing the elements of $\mathcal{P}$ as formally traceless symmetric polynomials, there are at least three other ways of writing them. As stated for certain R and ε they may be quotiented to form a matrix algebra. We may also write the elements of $\mathcal{P}$ in terms of Hahn polynomials and in terms of stereographic projection (see section 3.1). These alternative representations are useful for some of the proofs as well as leading to a greater understanding.

2.5.1 Finite dimensional representations

If $R^2 = \varepsilon^2(k(k+1))$ then we may quotient out all basis elements P_n^m, with $n \geq 2k+1$. This leaves just the $(2k+1)^2$ elements $\{P_n^m | n \leq 2k\}$. This algebra is exactly equivalent to the $(2k+1) \times (2k+1)$ matrix representation of $su(2)$.

$$\begin{aligned} J_0|k,j\rangle &= \varepsilon j|k,j\rangle \\ J_+|k,j\rangle &= \varepsilon(k-j)^{1/2}(k+j+1)^{1/2}|k,j+1\rangle \\ J_-|k,j\rangle &= \varepsilon(k+j)^{1/2}(k-j+1)^{1/2}|k,j-1\rangle \end{aligned}$$

This representation may be written in terms of Wigner's operators, see (4.3) below. The sesquilinear form is now positive definite[6], and given by the trace.

$$\langle f, g\rangle = \frac{1}{2k+1}\sum_{j=-k}^{k}\langle k,j|f^\dagger g|k,j\rangle$$

[6]From (2.8) one see that $\|P_n^m\|^2 > 0$ for $n < 2k+1$ and that $\|P_n^m\|^2 = 0$ otherwise

2.5.2 Writing P_n^m in terms of Hahn Polynomials

Given $f \in \mathcal{P}$ then we can use the commutation relations (2.3) to push the J_+ and J_- to the left of each term. If a J_+ and J_- appear in one term we can use the Casimir (2.4) identity to remove both. Thus the resulting terms must either have only J_+'s or only J_-'s or neither. If we collect all the terms with the same number of J_+ or J_- as their factors then f may be written as a sum of terms of the form

$$\{(J_+)^m p(z)\,,\, p(z)\,,\, (J_-)^{-m} p(z)\}$$

where $p(z)$ is a polynomial in z. Since P_n^m is a eigenvector of ad_z we have

$$P_n^m = \begin{cases} (J_+)^m(-\varepsilon)^{n-m}\begin{pmatrix} 2n \\ n-m \end{pmatrix}^{-1/2} h_{n-m}^{(m,m)}({}^{z}\!/\!_{\varepsilon} + \frac{N-1}{2}, N-m) & \text{for } m \geq 0 \\ (J_-)^{-m}(-1)^m(\varepsilon)^{n-m}\begin{pmatrix} 2n \\ n-m \end{pmatrix}^{-1/2} h_{n-m}^{(m,m)}({}^{z}\!/\!_{\varepsilon} - m + \frac{N-1}{2}, N-m) & \text{for } m < 0 \end{cases}$$

where $N^2 + 1 = 4R^2\varepsilon^{-2}$. Here $h_n^{(\alpha,\beta)}(x, N)$ is a Hahn Polynomial[7], following the notation of [12, chapter 2]. Examples of P_n^m for $n \leq 3$ written in this form are given in table 1.

3 The "Wick rotation" of $\mathcal{P}$ to the Hyperbolic case

The space of scalar functions on the sphere is a representation of $su(2)$. Now we consider extending this for a representation of $sl(2,\mathbb{C})$. This algebra contains both $su(2)$ and $su(1,1)$ as subalgebras. We can extend the results for sphere to the symmetric spaces of $su(1,1)$ viz the positive hyperboloid (hyperbolic disc) and negative hyperboloid, (two dimensional De-Sitter space).

Let us consider a new version of the algebra of noncommutative scalar functions given by:

$$\mathcal{P}(\varepsilon, R, \alpha) = \{\text{Polynomials in } J_+, J_-, J_0\} \Big/ = \tag{3.1}$$

where

$$[J_0, J_+] = \varepsilon J_+ \quad [J_0, J_-] = -\varepsilon J_- \quad [J_+, J_-] = 2\varepsilon\alpha^2 J_0 \quad J_0^2 + \frac{1}{2\alpha^2}(J_+J_- + J_-J_+) = R^2 \tag{3.2}$$

In the most general case $\varepsilon, R, \alpha \in \mathbb{C}$ are all independent and $\alpha \neq 0$. The hermitian conjugate is given by $J_0^\dagger = J_0$, $J_+^\dagger = J_-$ and $J_-^\dagger = J_+$, and the sesquilinear form is defined as before. By extending the results of [1] we have very similar results for the new basis elements P_n^m except with α's dotted about. A significant result is that for $\alpha^2 = -1$ and $R^2 < -\varepsilon^2$, the sesquilinear form of $\mathcal{P}(\varepsilon, R, \alpha)$ positive definite, making $\mathcal{P}(\varepsilon, R, \alpha)$ into a Hilbert space. However the operations on this spaces of multiplying by the elements J_0, J_+, J_- are not continuous. There must be some way of recovering the representation of $su(1,1)$. This is being investigated.

By using the Jordan-Schwinger representation of $sl(2,\mathbb{C})$ we should be able to generate $\psi_n^{(r,m)}$, to produce a theory of spinor and vector fields on the positive and negative hyperboloids.

[7] The Hahn polynomials are defined in a similar manner to Legendre polynomials, except one replaces the integral by a finite sum in the definition of the orthogonality property. In the limit $N \to \infty$ they tend to the Legendre polynomials

3.1 Stereographic projection for $\mathcal{P}$

The algebra $\mathcal{P}$ is equivalent to the algebra given by[8]

$$\mathcal{P}_z = \left\{ \text{Polynomials in } z, \overline{z}, \frac{1}{\widehat{R}^2 + \alpha^2 z\overline{z}} \right\} \Big/ \sim \tag{3.3}$$

where z and $\overline{z}$ are considered as independent and conjugate $z^\dagger = \overline{z}$, and $\widehat{R}^2 = R^2 + \frac{1}{4}\varepsilon^2$ as before. The quotient is given by

$$z\overline{z} - \overline{z}z \sim \frac{-\varepsilon}{8\widehat{R}^3\alpha^2}(4\widehat{R}^2 + \alpha^2 z\overline{z})(4\widehat{R}^2 + \alpha^2\overline{z}z) \tag{3.4}$$

For the case $R = 1$ and $\alpha^2 = -1$, this reduces to the formula given in [16]. This is equivalent to

$$\overline{z}z = \rho(z\overline{z}) \tag{3.5}$$

where ρ is the möbius transform

$$\rho(x) = \frac{(1 + \varepsilon/2\widehat{R})x + 2\varepsilon\widehat{R}/\alpha^2}{(-\varepsilon\alpha^2/8\widehat{R})x + (1 - \varepsilon/2\widehat{R})} \tag{3.6}$$

This möbius transform when written as a function of both ε and x satisfies $\rho^n(\varepsilon, x) = \rho(n\varepsilon, x)$ where $\rho^{n+1}(\varepsilon, x) = \rho(\varepsilon, \rho^n(\varepsilon, x))$.

We may consider this the analogue of stereographic projection. The relationship between the two algebras is given by (where $x = z\overline{z}$)

$$J_0 = \widehat{R}\frac{4\widehat{R}^2 - \alpha^2 x}{4\widehat{R}^2 + \alpha^2 x} - \frac{\varepsilon}{2} \qquad J_+ = i\overline{z}\frac{4\widehat{R}^2\alpha^2}{4\widehat{R}^2 + \alpha^2 x} \qquad J_- = -i\frac{4\widehat{R}^2\alpha^2}{4\widehat{R}^2 + \alpha^2 x}z \tag{3.7}$$

We note that if $f \in \mathcal{P}^m$ then the denominator of f is $(\widehat{R}^2 + \alpha^2 x)^n$, and the numerator is a polynomial of z, $\overline{z}$ and x. Thus the numerator of P_n^m will be a polynomial in z and x or $\overline{z}$ and x depending on the sign of m. This polynomial is related to the Hahn polynomials and its degree will be less than n.

There also exists projections for mapping the noncommutative negative hyperboloid to a noncommutative cylinder.

3.2 The commutative limit

When $\varepsilon = 0$ the algebra $\mathcal{P}$ reduces to the commutative algebra of functions on either the sphere or the hyperboloid depending on the sign of α^2. For the sphere and the positive hyperboloid the algebra $\mathcal{P}_z$ become the stereographic projection, whilst for the negative hyperboloid we have to consider the analytic continuation of the stereographic projection. In the case of the cones the stereographic projection is degenerate. See table for the four symmetric spaces of $su(2)$ or $su(1,1)$.

4 Spinors and Vectors on the Noncommutative Sphere

In [2] we consider how one can extend the algebra $\mathcal{P}(\varepsilon, R)$ to include vector and spinor fields. The result is a new algebra (Ψ, ρ), where Ψ is a set of polynomials, and ρ is a noncommutative

[8] here z and x are not the same as the z in section 2, z is replaced by J_0 and x by $\frac{1}{2}(J_+ + J_-)$

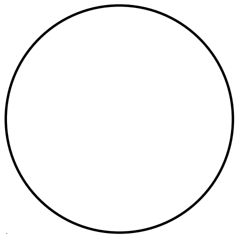
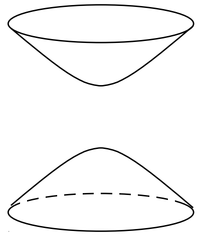
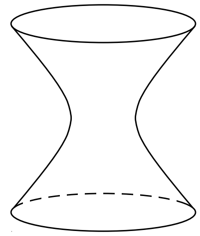
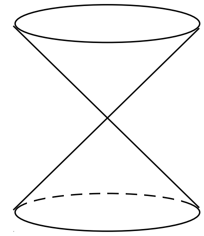
	Sphere	Positive hyperboloid Hyperbolic disc	Negative hyperboloid 2 dim De-Sitter space	Two cones
				
coordinate system	$\alpha^2 = 1$ $R^2 > 0$ $J_0 = R\cos\theta$ $J_+ = ie^{-i\phi}R\sin\theta$ $J_- = -ie^{-i\phi}R\sin\theta$ $0 \le \theta \le \pi$ $0 \le \phi < 2\pi$	$\alpha^2 = -1$ $R^2 > 0$ $J_0 = R\cosh\eta$ $J_+ = ie^{-i\phi}R\sinh\eta$ $J_- = -ie^{-i\phi}R\sinh\eta$ $0 \le \eta < \infty$ $0 \le \phi < 2\pi$	$\alpha^2 = -1$ $R^2 < 0$ $J_0 = iR\sinh\eta$ $J_+ = -e^{-i\phi}R\cosh\eta$ $J_- = e^{-i\phi}R\cosh\eta$ $-\infty < \theta < \infty$ $0 \le \phi < 2\pi$	$\alpha^2 = -1$ $R^2 = 0$ $J_0 = \eta$ $J_+ = ie^{-i\phi}\|\eta\|$ $J_- = -ie^{-i\phi}\|\eta\|$ $-\infty < \eta < \infty$ $0 \le \phi < 2\pi$
Stereographic coordinates $z = -ire^{i\phi}$, $z = ire^{-i\phi}$	$r = 2R\tan(\theta/2)$ $z \in \mathbb{C}$	$r = 2R\tanh(\theta/2)$ $z \in \mathbb{C}$, $\|z\| < 1$	$r = 2iR\tanh(\eta/2 - i\pi/2)$ z is not a complex variable	degenerate

Table 2: The two dimensional symmetric spaces of $sl(2,\mathbb{C})$

and nonassociative product on Ψ. There is a natural basis of Ψ given by $\{\psi_n^{(r,m)}\}$ where $n = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}\ldots$ and $m, r = -n, -n+1, \ldots, n$. It is useful to consider the subspace

$$\Psi_\bullet^{(r,\bullet)} = \text{span}\{\psi_n^{(r,m)} \mid \forall n, r\}$$

The subspace $\Psi_\bullet^{(0,\bullet)}$ together with the product ρ may be identified with $\mathcal{P}(\varepsilon, R)$, the space of scalar functions on the sphere. Thus $\psi_n^{(0,m)} = P_n^m$. This is analogous to (2.1). All other $\Psi_\bullet^{(r,\bullet)}$ are modules over $\Psi_\bullet^{(0,\bullet)}$ in an analogous way to the way spinors and vector fields are modules over scalar fields. The spaces $\Psi_\bullet^{(1,\bullet)}, \Psi_\bullet^{(-1,\bullet)}, \Psi_\bullet^{(1/2,\bullet)}$ will be identified with vector, covector, and spinor fields respectively.

In the limit, $\varepsilon = 0$, the basis element $\psi_n^{(r,m)}$ becomes the rotation matrix element $D_{mr}^n(\alpha, \beta\gamma)$. As a result we may view (Ψ, ρ) as the noncommutative and nonassociative analogue of the algebra of functions on the Lie group $SU(2)$.

4.1 The mechanics of setting up Ψ

The first two chapters of [2] concern themselves with the setting of (Ψ, ρ). For an outline, consider the Jordan-Schwinger representation of $su(2)$ given by

$$J_0 = \tfrac{1}{2}(a_+a_- - b_+b_-) \qquad J_+ = a_+b_- \qquad J_- = a_-b_+$$

where $[a_-, a_+] = \varepsilon$ and $[b_-, b_+] = \varepsilon$ are the generators of $\mathcal{W}$ the product of two Heisenberg-Weil algebras. The Casimir of this representation of $su(2)$ is given by

$$J_0^2 + \tfrac{1}{2}J_+J_- + \tfrac{1}{2}J_-J_+ = K_0^2 - \tfrac{1}{4}\varepsilon^2$$

where

$$K_0 = \tfrac{1}{2}(a_+a_- + b_+b_- + \varepsilon)$$

We would like to take the square root of this equation and consider quotienting $\mathcal{W}$ by the ideal generated by $K_0 \sim \widehat{R} = (R^2 + \frac{1}{4}\varepsilon^2)^{1/2}$. However, although K_0 commutes with any polynomial in $\{J_0, J_+, J_-\}$, it does not commute with all the elements in $\mathcal{W}$. Therefore the left ideal generated by $K_0 \sim \widehat{R}$ is not a two sided ideal and the corresponding quotient product is not associative.

Many results about the definition and basis elements $\psi_n^{(r,m)}$ are similar to the corresponding case for the scalar functions P_n^m, but with caution reflecting the nonassociative nature of (Ψ, ρ). They are not given here and the reader is invited to look at the article.

However, I will quote the product formula for the basis elements, since it is not given in the above paper, but will hopefully appear in [3]. The combination of two basis elements is given by

$$\rho(\psi_{n_1}^{(r_1,m_1)} \psi_{n_2}^{(r_2,m_2)}) = \sum_{n=n_{\min}}^{n=n_1+n_2} C_{m_1\, m_2\, m_1+m_2}^{n_1\, n_2\, n} R_{r_1\, r_2\, r_1+r_2}^{n_1 n_2 n} \psi_n^{(r_1+r_2, m_1+m_2)} \tag{4.1}$$

where $n_{\min} = \max(|n_1 - n_2|, |r_1 + r_2|, |m_1 + m_2|)$, $C_{m_1\, m_2\, m_1+m_2}^{n_1\, n_2\, n}$ is the Clebsh-Gordon coefficient, and the reduced matrix element $R_{r_1\, r_2\, r_1+r_2}^{n_1 n_2 n}$ is given by[9]

$$\begin{aligned} R_{r_1\, r_2\, r_1+r_2}^{n_1 n_2 n} = (-1)^{2k+n_1+n_2+r_1+r_2} & \frac{\|\psi_{n_1}^{(r_1,\bullet)}\|_{k+r_2} \|\psi_{n_2}^{(r_2,\bullet)}\|_k}{\|\psi_n^{(r_1+r_2,\bullet)}\|_k} (2k + 2r_2 + 1)^{1/2} \times \\ & (2n_1 + 1)^{1/2} (2n_2 + 1)^{1/2} \begin{Bmatrix} k + r_1 + r_2 & n_1 & k + r_2 \\ n_2 & k & n \end{Bmatrix} \end{aligned} \tag{4.2}$$

where the symbol in the curly brackets is Wigner's 6-j coefficient.

This is proved by considering $\psi_n^{(r,m)}$ as one of Wigner's operators given in [10, eqn (3.340)] as

$$\psi_n^{(r,m)} |k, j\rangle = (-1)^{n-r} \|\psi_n^{(r,\bullet)}\|_k \frac{(2n+1)^{1/2} (2k+1)^{1/2}}{(2k + 2r + 1)^{1/2}} \left\langle \begin{matrix} & n+r & \\ 2n & & 0 \\ & n+m & \end{matrix} \right\rangle |k, j\rangle \tag{4.3}$$

and then using the product law given by [10, eqn (3.350)]. In [10] the Wigner's operators are indeed referred to as discrete rotation matrices.

4.2 Physical interpretation

As mentioned in the introduction, the basis elements can be viewed as the nonassociative and noncommutative analogue of the rotation matrices. However they may also be interpreted in terms of vectors and spinors on the sphere. This interpretation is not as clear cut as the case for the scalars and I hope it is not the final word on the matter.

4.2.1 The exterior derivative, 1-forms, and vector fields

In standard geometry there are many equivalent definitions of a vector field. However not all of them can be extended to the noncommutative case at the same time. If ones take the definition of a vector field given that it must satisfy the Leibniz formula, and extends this definition to noncommutative geometry then the vector fields are given by $X = \mathrm{ad}_f$ for some $f \in \mathcal{P} = \Psi_\bullet^{(0,\bullet)}$. This can then be used to give a definition of the exterior derivative and the exterior algebra [4]. The problem with this definition is that this space does not form a module over the space of functions.

[9] Here $\|\psi_n^{(r,\bullet)}\|_k$ is given by substituting $R = \varepsilon(k(k+1))^{1/2}$ into the formula for $\|\psi_n^{(r,\bullet)}\|^2$ and taking the positive square root.

Here we give another definition of vectors which do form a module over the space of functions. We start by defining the covectors by use of the analogue of the following definition of the exterior derivative of scalar fields:

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i \tag{4.4}$$

If we consider the noncommutative sphere as a three dimensional manifold with normalised basis coordinates $x^m = (2\widehat{R} + \varepsilon)^{-1/2} \psi_1^{(0,m)}$ for $m \in \{-1, 0, 1\}$ we can define the basis 1-forms as

$$dx^m = (2\widehat{R} + \varepsilon)^{-1/2} \psi_1^{(-1,m)} \qquad \text{for } m \in \{-1, 0, 1\}$$

This enables us to define the *exterior derivative* on the space of functions as

$$d : \Psi_\bullet^{(0,\bullet)} \mapsto \Psi_\bullet^{(-1,\bullet)} \qquad\qquad df = \sum_{m=-1}^{1} (-1)^{m+1} dx^{-m} \mathrm{ad}_{x^m} f \tag{4.5}$$

which is analogous to (4.4). This is not a derivative but does satisfy the Leibniz rule in the limit as $\varepsilon \longrightarrow 0$

$$d(fg) = d(f)g + f\, d(g) + O(\varepsilon) \tag{4.6}$$

We can now define the *vectors fields* as the set $\Psi_\bullet^{(1,\bullet)}$. The action of a vector on a scalar is given by

$$X(f) = \rho((df)\, X)$$

By the same reasoning as (4.6), this is also only a derivation in the limit $\varepsilon \longrightarrow 0$.

$$X(fg) = X(f)g + fX(g) + O(\varepsilon)$$

One must therefore decide which of the properties of vector fields one wishes to extend to noncommutative geometry, since the property of being a derivative, and the property of being a module over the space of functions are incompatible.

We also observe that

$$\sum_{m=-1}^{1} x^m X_m = \sum_{m=-1}^{1} X_m x^m = 0$$

identically for all ε, where $X_m = (dx^m)^\dagger$. This is equivalent to requiring that $x^m(\partial/\partial x^m) = 0$, i.e. X^m are vector fields on a sphere.

We can now define a *metric*

$$g : \Psi_\bullet^{(1,\bullet)} \times \Psi_\bullet^{(1,\bullet)} \mapsto \Psi_\bullet^{(0,\bullet)} \qquad\qquad g(X, Y) = \rho(X^\dagger Y)$$

This can be extended for all spaces $\Psi_\bullet^{(r,\bullet)}$.

4.2.2 Problems with this interpretation

There are many problems with this interpretation of our system. Here is a list of cases where this noncommutative geometry is different from the commutative geometry even in the limit $\varepsilon = 0$:

(1) the space $\Psi_\bullet^{(-1,\bullet)}$ is the image of $\Psi_\bullet^{(0,\bullet)}$ under d. This means that all 1-forms are closed.

(2) The definition (4.5) can be extended as a map $d : \Psi_\bullet^{(r,\bullet)} \mapsto \Psi_\bullet^{(r-1,\bullet)}$ for all r. However this map does not satisfy $d^2 = 0$ even in the limit. As a result no exterior calculus is defined here.

(3) The set of vectors X_i are not dual, in the usual sense, to the set of 1-forms dx^j, or alternatively the vectors X_i are not orthogonal with respect to the metric g. This is because for $i \neq j$

$$g(X_i, X_j) = \rho(dx^i X_j) = X_j(x^i) \neq 0$$

for all ε even when $\varepsilon = 0$. However we do have $\pi_0(g(X_i, X_j)) = \delta_{ij}$

Some of these problems, together with the non derivative nature of d, may be solved by redefining the space of covectors. For example it may be similar to (4.7) below.

4.2.3 Spinor fields

It is natural now to define the set $\Psi_\bullet^{(1/2,\bullet)}$ as the space of spinor fields and its dual $\Psi_\bullet^{(-1/2,\bullet)}$. This means that a vector is the product of two spinors. Using the usual definition of rotation by 2π, we see that rotation by 2π does not change the sign of $\psi_n^{(r,m)}$ if r is an integer, i.e. for scalars, vectors and other "Bosons". Whilst $\psi_n^{(r,m)}$ changes sign under rotation of 2π for r a half integer, i.e. for spinors fields and other "Fermions".

An alternative way to define spinor fields is as the set

$$\mathcal{S} = \left\{ \begin{pmatrix} a_+ \\ b_+ \end{pmatrix} f_1 + \begin{pmatrix} a_- \\ b_- \end{pmatrix} f_2 \;\middle|\; f_1, f_2 \in \Psi_\bullet^{(0,\bullet)} \right\} \tag{4.7}$$

This is decomposed into $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$ corresponding to the eigenspaces of ad_{K_0}, which is now regarded as the chirality operator. This interpretation is now equivalent to the one proposed by Grosse et al [7, 8, 9], who go on to define and solve the Dirac equation. We note that $\mathcal{S}$ is not equivalent to simply two copies of $\Psi_\bullet^{(-1/2,\bullet)} \oplus \Psi_\bullet^{(1/2,\bullet)}$, but a proper subset. This is because the element $\begin{pmatrix} a_+ \\ 0 \end{pmatrix} \notin \mathcal{S}$. In the commutative limit $\varepsilon = 0$ we can obtain the standard spinors on a sphere.

5 Outlook: A Path Towards Quantum Gravity

Figure 1 shows how the articles described fit into my personal quest for quantum gravity.

(1) Paper [1] as described in section 2 has three natural extensions $\{(2), (3), (4)\}$

(2) Paper [2] as described in section 3. One can also consider the noncommutative supersphere, which is a representation of the superalgebra $osp(2, 1)$. Thus one can study possible analogies of the Dirac operator.

(3) It was possible to give a noncommutative version of the sphere because it was a co-adjoint orbit. By using other groups one could extend these results to higher dimensional compact co-adjoint orbits. One could consider the algebra generated by the quotienting the enveloping algebra of a Lie algebra by the set of equations equivalent to the Casimirs. As a long term goal one must consider noncommutative analogous of non co-adjoint orbits.

(4) Paper [3] (hopefully) as described in section 4.

(5) The work by Madore [4] and Coquereaux [14] examine an algebraic definition connection which can be used in noncommutative geometry. It requires simply that the vector space of vector fields is a module over the space of functions. This is the case in (2).

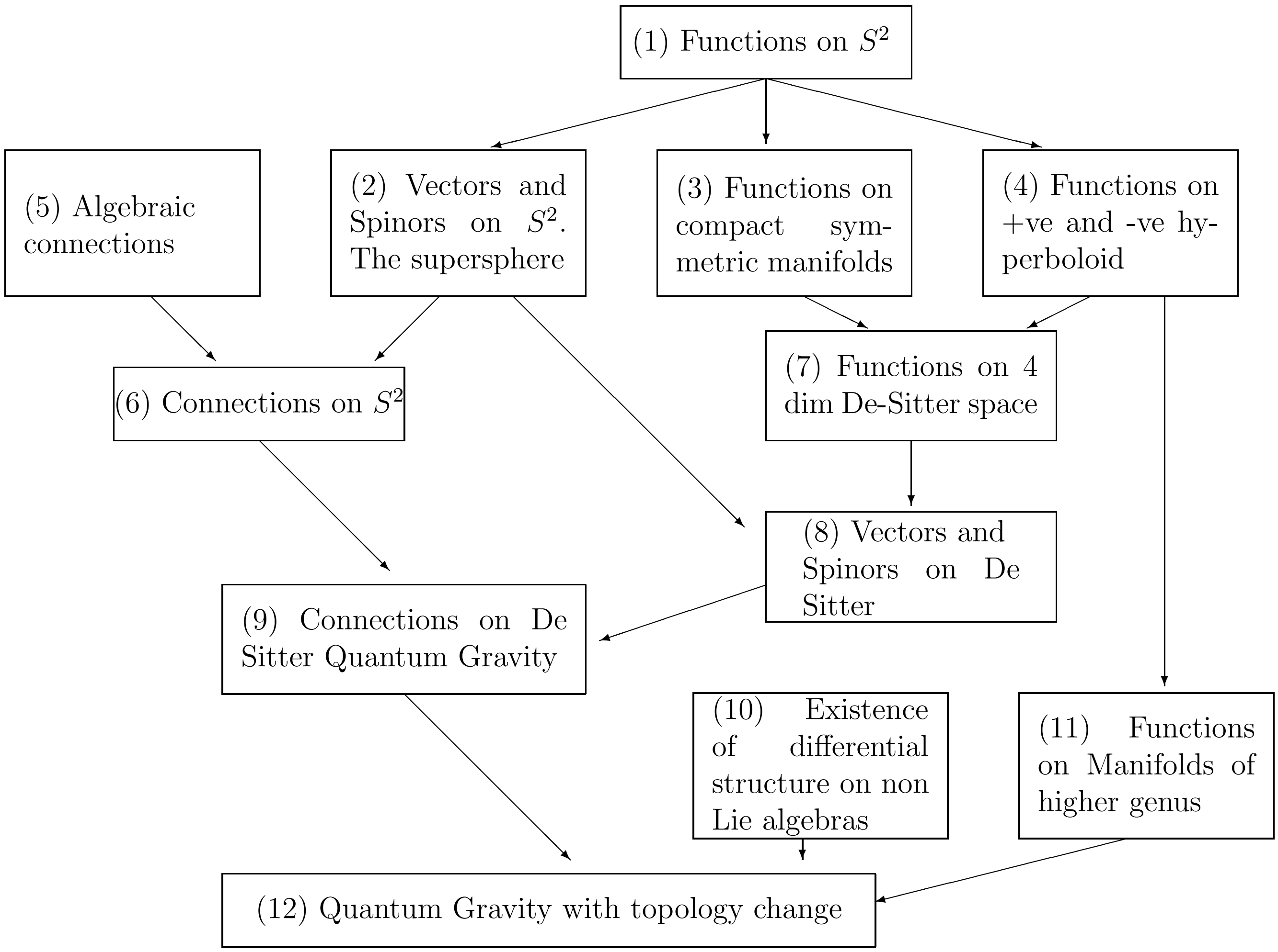


Figure 1: My path towards Quantum gravity

(6) As stated the work in (5) is applicable to (2).

(7) By combining (3) and (4) we can generate an algebra for the 4 dimensional De-Sitter space. This would give this spacetime a natural cellular structure and enable one to study scalar fields on this space.

(8) The Jordan-Schwinger algebra can be generated for any Lie algebra. It may be possible therefore to extend the results of (2) to generate an algebra for spinors and vector fields on the 4 dimensional De-Sitter space. Since this will enable one to look at spinor and vector fields on a space with a natural cellular structure, it might give an alternative method of renormalisation.

(9) It may be possible to put a (non metric) connection on the noncommutative 4-dimensional De-Sitter universe and thus have a version of quantum gravity.

(10) Work by myself [13] and Madore et al [15] have looked at the algebraic structures that are required in order to be able to define a exterior differential structure

(11) Klimek and Lesniewski [17] have recently given an outline of how one can create noncommutative analogues of 2-dimensional manifolds with genus greater than 2. They use the noncommutative disc, so any new results for that should be useful.

(12) It is believed that the true version of quantum gravity will naturally lead to topology change.

References

[1] J Gratus. A Natural Basis of States for the Noncommutative Sphere and its Moyal Bracket J. Maths. Phys **38** (8), (Aug 1997) 4283 - 4300, q-alg/9703038

[2] J Gratus. A Natural Basis of Vector and Spinor States for the Noncommutative Sphere. Submitted to J. Maths. Phys, August 1997 q-alg/9708003

[3] J Gratus. The Noncommutative Hyperboloids. Possibly a letter. Being written

[4] J. Madore 1995, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, Cambridge University Press.

[5] J. Madore, *The Commutative Limit of a Matrix Geometry.* J. Math. Phys. **32**(2), 332-335 (1991)

[6] H. Grosse, C. Klimčík, P. Prešnajder *Finite Quantum Field Theory in Noncommutative Geometry* Int. J. Theo. Phys. **35**(2), 231-244 (1996)

[7] H. Grosse, C. Klimčík, P. Prešnajder, *Topologically Nontrivial Field Configurations in Noncommutative Geometry.* Comm. Math. Phys. **178**(2), 507-526 (1996)

[8] H. Grosse, P. Prešnajder, *The Dirac Operator on the Fuzzy Sphere* Lett. Math. Phys. **33**(2) 171-181 (1995)

[9] H. Grosse, C. Klimčík, P. Prešnajder, *Field Theory on a Supersymmetric Lattice* Comm. Math. Phys. **185**, 155-175 (1997)

[10] L. C. Biedenharn J.D. Louck Angular momentum in Quantum Physics: Theory and Application Encyclopedia of Mathematics and its Applications, Volume 8 Addison-Wesley 1981

[11] L. C. Biedenharn J.D. Louck The Racah-Wigner Algebra in Quantum Theory Encyclopedia of Mathematics and its Applications, Volume 9 Addison-Wesley 1981

[12] A. F. Nikiforov, S. K. Suslov, V. B. Uvarov 1991, *Classical Orthogonal Polynomials of a Discrete Variable,* Springer-Verlag

[13] J Gratus. Non Commutative Differential Geometry, and the matrix Representations of Generalised Algebras. *Journal of Geometry and Physics.* To appear July 1997 q-alg/9703034

[14] R. Coquereaux, *Algebraic Connections on Parallel Universes* Int. J. Mod. Phy A **10**, (1) 89-98 (1995)

[15] Dimakis A., Madore J., Mourad J. *Quantum Space-time and Classical Gravity* Preprint Orsay 1996,

[16] S. Klimek, A. Lesniewski, *Quantum Riemann Surfaces: I The Unit Disc* Commun. Math. Phys. **146** 103-122, 1992

[17] S. Klimek, A. Lesniewski, *Quantum Riemann Surfaces: II The Discrete Series* Lett. Math. Phys. **24** 125-139, 1992